

**15.053**

**February 22, 2007**

- **Introduction to the Simplex Algorithm**

## **Quotes for today**

**Give a man a fish and you feed him for a day. Teach him how to fish and you feed him for a lifetime.**

**-- Lao Tzu**

**Give a man a fish dinner, and he will forget it by next week. Let a person catch the fish for himself, and he'll remember it for a lifetime.**

**-- Jim Orlin**

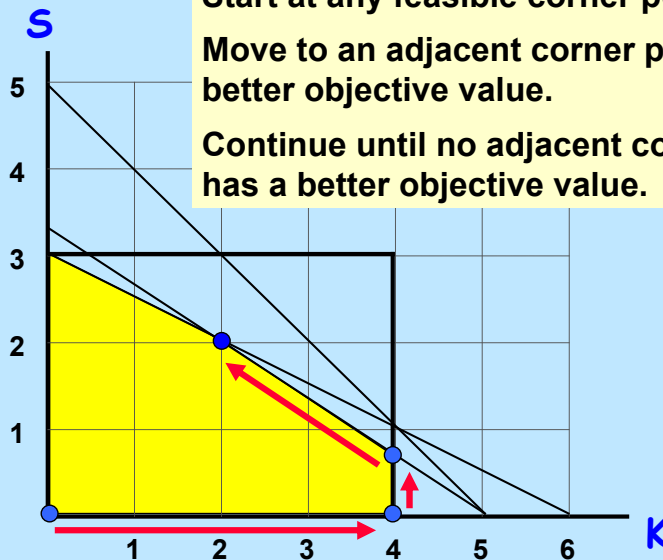
## Preview of the Simplex Method

Maximize  $z = 3K + 5S$

Start at any feasible corner point.

Move to an adjacent corner point with better objective value.

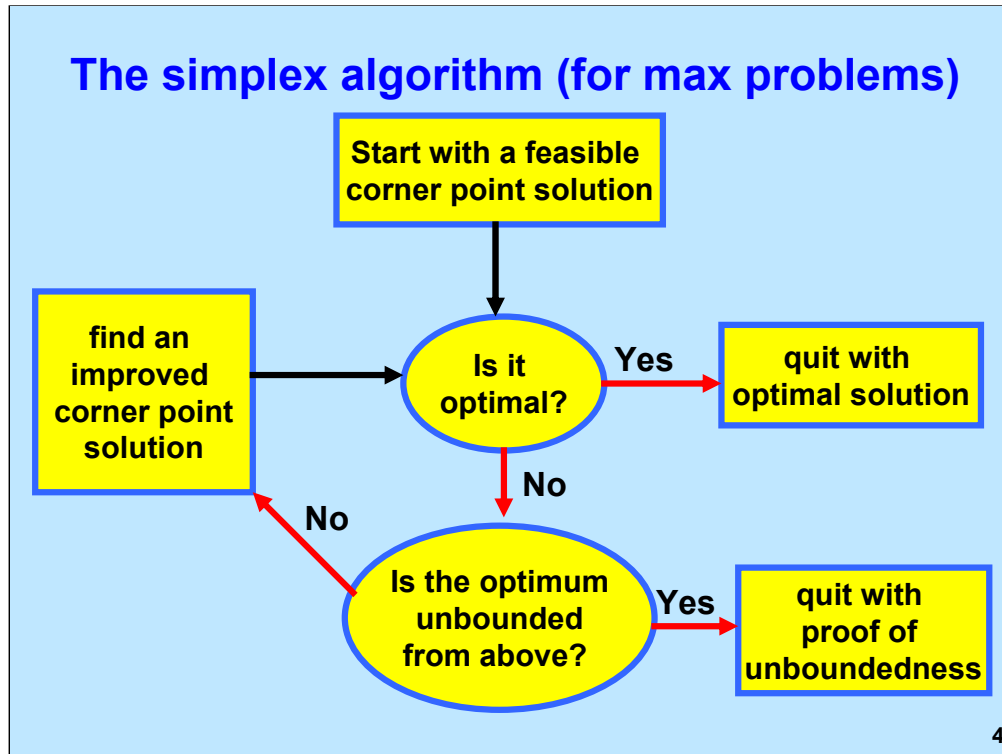
Continue until no adjacent corner point has a better objective value.



3

This is a picture of the simplex algorithm in inequality form. In this form, the simplex algorithm moves from corner point to corner point. And each corner point is the intersection of two constraints.

When we move to equality form, the simplex algorithm still moves from corner point to corner point. And the corner points are still found by solving a system of equations. So, there are many similarities.



As you can see, this is a fairly simple structure. At the same time, it may be difficult to keep everything in one's head at the same time. That is where the two dimensional example can help out.

We will assume that we start with a feasible corner point solution. That immediately raises two questions. What does a corner point solution look like? And how do you find a corner point solution to start with? Both of these issues will be addressed shortly.

The next slides deal with something even more preliminary. We will be assuming that we start with a linear program with equality constraints and non-negativity constraints, and nothing else. So we need to get each linear program into the correct starting form. We will show how to do that on the next few slides.

## Goals for this lecture

### Major Issues of the Simplex Algorithm

1. How does one get the LP into the correct starting form?
2. How does one recognize optimality and unboundedness?
3. How does one move to the next corner point solution?

**Note: we will derive the simplex algorithm in class!**

## Linear Programs in Standard Form

We say that a linear program is *in standard form* if the following are all true:

1. Non-negativity constraints for all variables.
2. All remaining constraints are expressed as equality constraints.
3. The right hand side vector,  $b$ , is non-negative.

### An LP not in Standard Form

$$\begin{array}{ll} \text{maximize} & z = 3x_1 + 2x_2 - x_3 + x_4 \\ & x_1 + 2x_2 + x_3 - x_4 \leq 5; \quad \text{not equality} \\ & -2x_1 - 4x_2 + x_3 + x_4 \leq -1; \quad \text{not equality} \\ & x_1 \geq 0, x_2 \geq 0 \quad x_3 \text{ and } x_4 \text{ may be negative} \end{array}$$

6

Excel Solver does not require that you write an LP in standard form because it will immediately transform it to standard form via software. We show next what linear programming solvers do with an LP that does not start in standard form.

## Converting Inequalities into Equalities Plus Non-negatives

**Before**

$$x_1 + 2x_2 + x_3 - x_4 \leq 5$$

**After**

$$x_1 + 2x_2 + x_3 - x_4 + s_1 = 5$$

$$s_1 \geq 0$$

$s_1$  is called a **slack variable**, which measures the amount of “unused resource.”

Note that  $s_1 = 5 - x_1 - 2x_2 - x_3 + x_4$ .

To convert a “ $\leq$ ” constraint to an equality, add a slack variable.

7

So, we transform a “ $\leq$  constraint” by

1. adding a slack variable
2. requiring that the slack variable is non-negative.

## Converting RHS and “ $\geq$ ” constraints

- Consider the inequality  $-2x_1 - 4x_2 + x_3 + x_4 \leq -1$ ;
- Step 1. Eliminate the negative RHS. Multiply by -1.

$$2x_1 + 4x_2 - x_3 - x_4 \geq 1$$

- Step 2. Convert to an equality

$$2x_1 + 4x_2 - x_3 - x_4 - s_2 = 1$$

$$s_2 \geq 0$$

- The variable added will be called a “*surplus variable*.”

To convert a “ $\geq$ ” constraint to an equality, subtract a surplus variable.

8

We get rid of negative right hand sides by multiplying through by -1.

We transform a “ $\geq$  constraint” by

1. adding a surplus variable
2. requiring that the slack variable is non-negative.

To be honest, I sometimes confuse the names “slack” and “surplus” because they are serving the exact same function, converting an inequality constraint to an equality constraint. They have different names because of their interpretations in practice. Often a “ $\leq$  constraint” will model a case in which we have limited resources, and the “slack” represents the amount left over. Often a “ $\geq$  constraint” will model a case in which we have to produce at least a specified amount. If we produce more than we need, we are said to have produced a surplus.



## Converting Max to Min and Min to Max

**Converting Max to Min: multiply objective by -1**

**Example: Minimize  $z = 3x_1 + 2x_2$   
subject to "constraints"**

**Has the same optimum solution(s) as**

**Maximize  $v = -3x_1 - 2x_2$   
subject to "constraints"**

9

Minimizing  $z$  is equivalent mathematically to maximizing  $-z$ . Interestingly, practitioners often have a very strong preference. If you tell a practitioner that you are maximizing the negative of the cost, it will sound very confusing, unless you convert it somehow to maximizing profit. But mathematically, there is no important distinction.

## **Other transformations**

**See tutorial on transformations.**

## **Why standard form?**

**The simplex method is designed for problems with equality constraints and non-negativity constraints.**

10

The tutorial covers situations in which a variable  $x$  does not start with the constraint  $x \geq 0$ . It is possible that in a model, some variables are constrained to be non-positive, and possibly other variables have no constraint on sign at all. In all of these cases, the LP solver will first create an equivalent program in which all variables are constrained to be non-negative.

## Review: solving a system of Equations

$$2x_1 + 2x_2 + x_3 = 9$$

$$2x_1 - x_2 + 2x_3 = 6$$

$$x_1 - x_2 + 2x_3 = 5$$

	$x_1$	$x_2$	$x_3$	RHS
Equation 1	2	2	1	= 9
Equation 2	2	-1	2	= 6
Equation 3	1	-1	2	= 5

The set of equations with the  $x$ 's written in the top row is called a tableau. We will use tableaus to illustrate the simplex algorithm.

	$x_1$	$x_2$	$x_3$		RHS
Equation 1	1	1	1/2	=	9/2
Equation 2	0	-3	1	=	-3
Equation 3	0	-2	3/2	=	1/2

We want column 1 to be

1
0
0

Divide through equation 1 by 2.

Subtract two times equation 1 from equation 2.

Subtract equation 1 from equation 3.

12

For more information on solving systems of equations, see the tutorial on the website.

	$x_1$	$x_2$	$x_3$	=	RHS
Equation 1	1	0	5/6	=	7/2
Equation 2	0	1	-1/3	=	1
Equation 3	0	0	5/6	=	5/2

We want column 2 to be

0
1
0

Divide through equation 2 by -3.

Subtract equation 2 from equation 1.

Add two times equation 2 to equation 3.

	$x_1$	$x_2$	$x_3$		RHS
Equation 1	1	0	0	=	1
Equation 2	0	1	0	=	2
Equation 3	0	0	1	=	3

We want column 3 to be

0
0
1

Divide through equation 3 by  $5/6$ .

Subtract equation 3 from equation 1.

Add  $1/3$  times equation 3 to equation 2.

	$x_1$	$x_2$	$x_3$		RHS
Equation 1	1	0	0	=	1
Equation 2	0	1	0	=	2
Equation 3	0	0	1	=	3

**Resulting equations**  $x_1 = 1, x_2 = 2, x_3 = 3.$

**The solution is now obvious.**

**The system of equations is in a very special form.**

At the end, each column for a variable has a single 1 and two 0s.

The equations themselves are the same as the solution.

## 1. Start with a feasible corner point solution

- Start with a tableau in “***canonical form***”
  - LP has equality constraints and non-negativity constraints.
  - There is one “***basic***” variable for each equality constraint.
  - The column for the basic variable for constraint  $j$  has a 1 in constraint  $j$  and 0's elsewhere.
  - The remaining variables are called ***non-basic***.

16

Standard form does not necessarily give a corner point solution. But standard form is a good place to get started.

For a corner point solution,



## A "Tableau" in canonical form.



z	x <sub>1</sub>	x <sub>2</sub>	x <sub>3</sub>	x <sub>4</sub>	x <sub>5</sub>	
1	0	2	0	0	1	= 2
0	0	2	1	0	-1	= 4
0	0	-1	0	1	2	= 1
0	1	6	0	0	3	= 3

The non-basic variables are  $x_2$  and  $x_5$ .  
 $z$  is considered to be a basic variable.

If we got rid of the non-basic variables (as in erasing the columns for  $x_2$  and  $x_5$ ), then the resulting equations would be the same as the solution. That is, the equations would be  $x_3 = 4$ ,  $x_4 = 1$ ,  $x_1 = 3$ . In reality, we don't erase the columns. We just set the non-basic variables to 0, which is mathematically equivalent.

## The “basic feasible solution” or bfs

The basic variables are  $x_1, x_3, x_4,$  and  $z$

The non-basic variables are  $x_2, x_5$

Set the  
non-basic  
variables  
to 0

$z$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	
1	0	2	0	0	1	= 2
0	0	2	1	0	-1	4
0	0	-1	0	1	2	= 1
0	1	6	0	0	3	= 3

The basic feasible solution (bfs) is:

$$x_2 = x_5 = 0; \quad x_1 = 3, x_3 = 4, x_4 = 1, z = 2$$

18

We will use the term “basic feasible solution” or “bfs” throughout the rest of the semester. Every bfs is also a corner point solution, in that it is not the midpoint of a line segment joining two other solutions.

The simplex method will move from corner point to corner point along edges.

## When is a basic feasible solution (bfs) optimal?

Together we will derive the optimality conditions

### An example

$$\begin{array}{ll} \text{maximize} & z = -2x_2 - x_5 + 2 \\ \text{subject to} & x_1 = 3, x_3 = 4, x_4 = 1 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

What is an optimal solution for this problem?

19

The first example is an LP in which

1. The objective function only has terms for the nonbasic variables.
2. The coefficients of the variables in the objective function are nonpositive and only involve the nonbasic variables.
3. The only constraints on the non-basic variables are nonnegativity constraints.

So, all one needs to do is to set  $x_2$  and  $x_5$  optimally, which in this case sets them both to 0.

### A second example

$$\begin{array}{ll} \text{maximize} & z = -2x_2 - x_5 + 2 \\ \text{subject to} & x_1 = 3 - 6x_2 - 3x_5 \\ & x_3 = 4 - 2x_2 + x_5 \\ & x_4 = 1 + x_2 - 2x_5 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

**What is an optimal solution for this problem?**

20

The second example is an LP in which

1. The objective function only has terms for the nonbasic variables.
2. The coefficients of the variables in the objective function are nonpositive and only involve the nonbasic variables.
3. Setting the nonbasic variables to 0 gives a feasible solution.

In this case, setting the nonbasic variables to 0 gives a feasible solution with  $z = 2$ . And any other solution has  $x_2 \geq 0$  and  $x_5 \geq 0$ , and thus  $z \leq 2$ . So, the solution with the nonbasic variables set to 0 must be optimal.

So, all one needs to do is to set  $x_2$  and  $x_5$  optimally, which in this case sets them both to 0.

**When are sufficient conditions for a solution to be optimal?**

$$\begin{aligned} \text{maximize} \quad & z = -2x_2 - x_5 + 2 \\ \text{subject to} \quad & x_1 = 3 - 6x_2 - 3x_5 \\ & x_3 = 4 - 2x_2 + x_5 \\ & x_4 = 1 + x_2 - 2x_5 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

**A solution  $x_1, x_2, x_3, x_4, x_5$  is guaranteed to be optimal for an LP with non-negativity constraints whenever .....**

21

The objective function has the following properties:

1. The coefficients of the nonbasic variables are nonpositive
2. The coefficients of the basic variables are 0.

And the feasible solution  $x$  is obtained by setting the nonbasic variables to 0.

## Recognizing an Optimal bfs: Tableau Version

The basic feasible solution (bfs) is:

$$x_2 = x_5 = 0; \quad x_1 = 3, x_3 = 4, x_4 = 1, z = 2$$

It is optimal!

z	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	
1	0	2	0	0	1	= 2
0	0	2	1	0	-1	= 4
0	0	-1	0	1	2	= 1
0	1	6	0	0	3	= 3

maximize  $z = -2x_2 - x_5 + 2$  s.t.  $x \geq 0$

The opt solution is  $z = 2$ .

22

In the tableau form, the objective is written as

$$z + 2x_2 + x_5 = 2.$$

**Optimality conditions for a bfs in tableau form:** the coefficients in the z-row nonnegative for the nonbasic variables.

Note that tableaus that correspond to bfs's already have the following properties:

1. The coefficients of the basic variables in the objective function are 0
2. There is a feasible solution obtained by setting the nonbasic variables to 0.

Thus the optimality condition stated above for a bfs in tableau form are the same as from the previous slides.

## Optimality Conditions

z	x <sub>1</sub>	x <sub>2</sub>	x <sub>3</sub>	x <sub>4</sub>	x <sub>5</sub>	
1	0	2	0	0	1	= 2
0	0	2	1	0	-1	= 4
0	0	-1	0	1	2	= 1
0	1	6	0	0	3	= 3

maximize  $z = -2x_2 - x_5 + 2$

### Important Fact.

If there is no negative coefficient in the z row, the basic feasible solution is optimal!

Is the optimum unbounded from above?

Together we will derive the conditions for unboundedness.

### An example

$$\begin{array}{ll} \text{maximize} & z = -2x_2 + x_5 + 2 \\ \text{subject to} & x_1 = 3, x_3 = 4, x_4 = 1 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

What is an optimal solution for this problem?

24

The objective function (for a max problem) in this example satisfies the following conditions:

1. The coefficients of the basic variables in the objective are 0
2. There is a positive coefficient in the objective for a nonbasic variable
3. The only constraints on the nonbasic variables are nonnegativity constraints.

In this case, we can get a sequence of increasingly better solutions by making  $x_5$  increasingly larger.



### A second example

$$\begin{aligned} \text{maximize} \quad & z = -2x_2 + x_5 + 2 \\ \text{subject to} \quad & x_1 = 3 - 6x_2 + 3x_5 \\ & x_3 = 4 - 2x_2 + x_5 \\ & x_4 = 1 + x_2 + 2x_5 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

What is an optimal solution for this problem?

25

The objective function (for a max problem) in this example satisfies the following conditions:

1. The coefficients of the basic variables in the objective are 0
2. There is a positive coefficient in the objective for the nonbasic variable  $x_5$ .
3. For any fixed choice of  $x_5 > 0$ , there is a feasible solution in which the only positive variables are  $x_5$  and the current basic variables.

In this case, we can get a sequence of increasingly better solutions by making  $x_5$  increasingly larger.

### Directions of Unboundedness

$$\begin{aligned}
 \text{maximize} \quad & z = -2x_2 + x_5 + 2 \\
 \text{subject to} \quad & x_1 = 3 - 6x_2 + 3x_5 \\
 & x_3 = 4 - 2x_2 + x_5 \\
 & x_4 = 1 + x_2 + 2x_5 \\
 & x_1, x_2, x_3, x_4, x_5 \geq 0
 \end{aligned}$$

Let  $x_5 = \Delta$ . Let  $x_2 = 0$ .

Assume that  $\Delta \geq 0$

$$\begin{aligned}
 \text{Then} \quad & x_1 = 3 + 3\Delta \\
 & x_3 = 4 + \Delta \\
 & x_4 = 1 + 2\Delta \\
 & z = \Delta
 \end{aligned}$$

Direction of unboundedness

$$\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 4 \\ 1 \\ 0 \end{pmatrix} + \Delta \begin{pmatrix} 3 \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

26

When the solution is unbounded from above, we often keep track of the sequence of solutions whose objective is unbounded from above. This can be done very efficiently by storing a feasible  $x'$  solution and a direction of unboundedness  $y'$ . Then for every value of  $\Delta$ , the solution  $x' + \Delta y'$  is feasible. As  $\Delta$  gets increasingly larger, the objective for  $x' + \Delta y'$  gets increasingly larger, and approaches infinity in the limit.

## More on Directions of Unboundedness

A vector  $y$  is called a **direction of unboundedness** for a maximization problem if

1. For all feasible solutions  $x$  and all positive numbers  $\Delta$ , the vector  $x + \Delta y$  is feasible.
2. The objective value for  $y$  is positive.

**Fact:** an LP is unbounded from above if and only if there is a feasible solution and there is also a direction of unboundedness.

27

The property of direction of unboundedness is true for linear programs, but is not true for non-linear programs. For example, one could imagine a feasible region in two dimensions that is a spiral, and that the objective goes to infinity as one moves along the spiral. But there is no direction of unboundedness as defined on the slide.

## Unboundedness: Tableau Version

z	x <sub>1</sub>	x <sub>2</sub>	x <sub>3</sub>	x <sub>4</sub>	x <sub>5</sub>	
1	0	2	0	0	-1	= 2
0	0	2	1	0	-1	= 4
0	0	-1	0	1	-2	= 1
0	1	6	0	0	-3	= 3



A maximization LP is unbounded from above if there is a bfs and a non-basic variable  $x_s$  such that

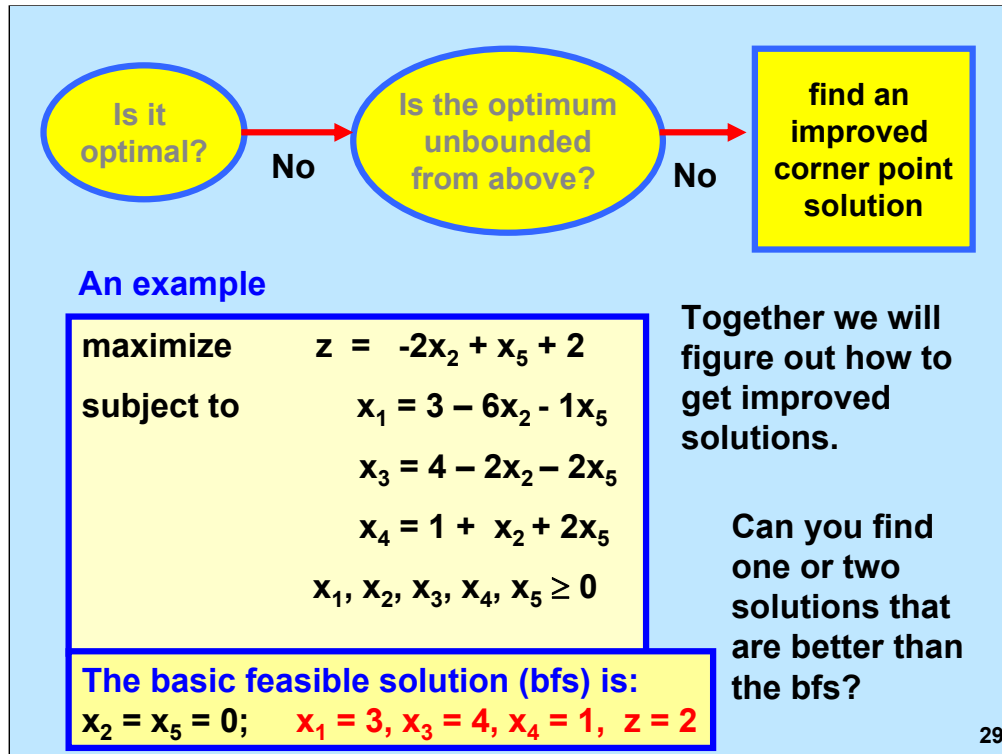
1. The coefficient for  $x_s$  in the z-row is negative, and
2. All coefficients in the column for  $x_s$  are  $\leq 0$ .

28

In the tableau form, the objective is written as  
 $z + 2x_2 - x_5 = 2$ .

**Unboundedness conditions when given a bfs in tableau form for a max problem:** there is a negative coefficient in the z-row for some nonbasic variable  $x_s$ .  
 The column in the tableau for  $x_s$  is nonpositive.

For any specified value of  $x_s$ , one can adjust the values of the current basic variables to provide a feasible solution. One shows that the objective value is unbounded from above by letting  $x_s$  approach infinity.



In this example, one of the basic variables  $x_5$  has a positive coefficient in the objective function. But the unboundedness conditions are not satisfied.

If we make  $x_5$  a little larger than 0, we can adjust the current basic variables to give a feasible solution and this feasible solution will have a larger objective value than the current bfs.

The larger that  $x_5$  is, the larger will be the objective value. So, we want to make  $x_5$  as large as possible so long as the other basic variables remain non-negative.

## Finding improved solutions

$$\max z = -2x_2 + x_5 + 2$$

$$\text{st} \quad x_1 = 3 - 6x_2 - 1x_5$$

$$x_3 = 4 - 2x_2 - 2x_5$$

$$x_4 = 1 + x_2 + 2x_5$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

We copied the equations so that there would be space to write the improved solutions.

## Improving Solutions: Tableau Version

z	x <sub>1</sub>	x <sub>2</sub>	x <sub>3</sub>	x <sub>4</sub>	x <sub>5</sub>		
1	0	2	0	0	-1	=	2
0	0	2	1	0	2	=	4
0	0	-1	0	1	-2	=	1
0	1	6	0	0	1	=	3

$$z = \Delta + 2$$

$$x_1 = 3 - 1\Delta$$

$$x_2 = 0$$

$$x_3 = 4 - 2\Delta$$

$$x_4 = 1 + 2\Delta$$

$$x_5 = \Delta$$

**Find a non-basic variable with a negative coefficient in the z-row. Set that variable to  $\Delta$ , and keep all other non-basic variables at 0.**

**Choose  $\Delta$  maximum**

We could look for improved solutions by just guessing the value of  $x_5$ . But to do it systematically, we set it to  $\Delta$ . As you can see, I am fond of using  $\Delta$  as a parameter.

Once we set it to  $\Delta$ , we can see how the current basic variables vary as a linear function of  $\Delta$ . We then choose  $\Delta$  as high as possible so that all of the current basic variables are nonnegative. In this case, we can let  $\Delta$  be as large as 2. If it were any larger, then  $x_3$  would be negative.

## **Mira and Marnie's M&M Adventure**

**Mira and Marnie, two MIT undergraduates known as the M&M sisters, recently received a gift from their parents of 2000 pounds of gray M&Ms and 6000 pounds of red M&Ms, the MIT colors. So, they decided to go into business selling large bags of "MIT M&Ms" for frat parties. They can sell a bag with 3 pounds of red M&Ms and 2 pounds of gray M&Ms for \$20. They can purchase bags of 3 pounds of red M&Ms and 4 pounds of gray M&Ms for \$30. How many bags should Mira & Marnie buy and sell to maximize their profit.**

32

M&Ms really can be bought in very large packages with quantity discounts, and you can choose the colors. You can even have custom printing (e.g., I love 15.053). See <http://www.mymms.com> for more details.



## Formulation as a linear program

- Let  $x_1$  be the number of 7 pound bags purchased (in thousands)
- Let  $x_2$  be the number of 5 pound bags sold (in thousands)
- Measure the profit in \$10,000s.

## A 2-variable LP

<u>maximize</u>	$z =$	$-3x_1 + 2x_2$	
<u>subject to</u>		$-3x_1 + 3x_2$	$\leq 6$
		$-4x_1 + 2x_2$	$\leq 2$
		$x_1 \geq 0, x_2 \geq 0$	

<u>maximize</u>	$z =$	$-3x_1 + 2x_2$	
<u>subject to</u>		$-3x_1 + 3x_2 + x_3$	$= 6$
		$-4x_1 + 2x_2 + x_4$	$= 2$
		$x_1, x_2, x_3, x_4 \geq 0$	

z	x <sub>1</sub>	x <sub>2</sub>	x <sub>3</sub>	x <sub>4</sub>	=	
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

We first add slack variables  $x_3$  and  $x_4$ .

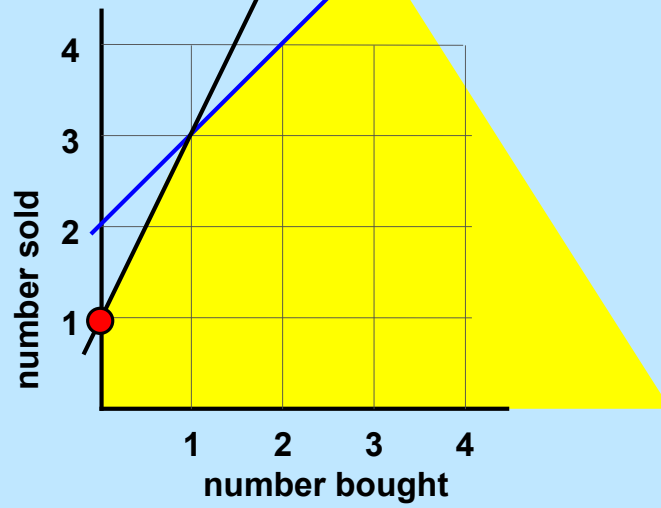
We then express the equations in tableau form.

Note that the initial tableau is in canonical form, and there is a corresponding bfs.

## The two dimensional geometry

$$-3x_1 + 3x_2 + x_3 = 6$$

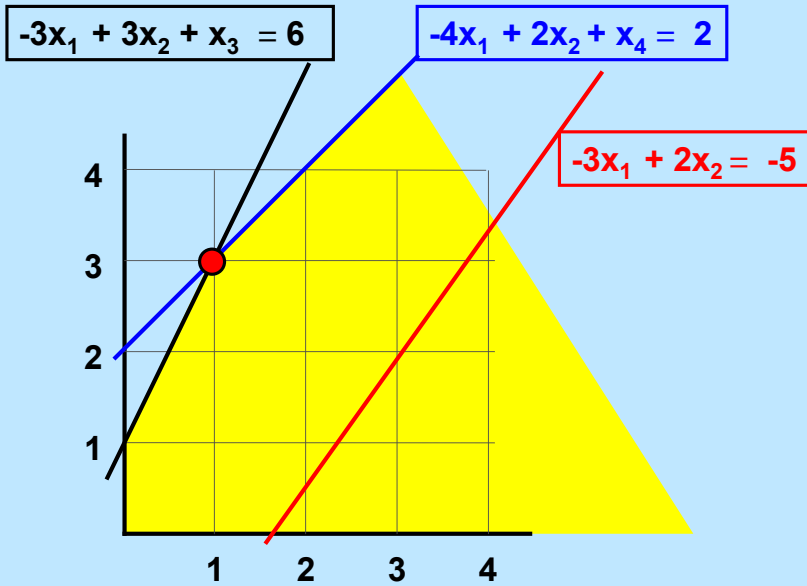
$$-4x_1 + 2x_2 + x_4 = 2$$



35

For this particular LP, the feasible region is unbounded, but there will be an optimal solution.

## The two dimensional geometry



36

The optimal solution will be  $x_1 = 1$  and  $x_2 = 3$ . The slack variables will both be 0.

## LP “canonical form”

The initial tableau is already in canonical form.

	z	$x_1$	$x_2$	$x_3$	$x_4$		
	1	3	-2	0	0	=	0
	0	-3	3	1	0	=	6
	0	-4	2	0	1	=	2

The **basic variables** are z,  $x_3$  and  $x_4$ .

The **non-basic variables** are  $x_1$  and  $x_2$ .

The **basic feasible solution** (bfs) for this basis is  
 $z = 0, x_1 = 0, x_2 = 0, x_3 = 6, x_4 = 2$

## LP Canonical Form and the bfs.

<b>z</b>	<b>x<sub>1</sub></b>	<b>x<sub>2</sub></b>	<b>x<sub>3</sub></b>	<b>x<sub>4</sub></b>		
<b>1</b>	<b>3</b>	<b>-2</b>	<b>0</b>	<b>0</b>	=	<b>0</b>
<b>0</b>	<b>-3</b>	<b>3</b>	<b>1</b>	<b>0</b>	=	<b>6</b>
<b>0</b>	<b>-4</b>	<b>2</b>	<b>0</b>	<b>1</b>	=	<b>2</b>

The text treats z as a basic variable.

The simplex method starts with a *tableau* in **LP canonical form** (or it creates canonical form at a preprocess step.)

The first solution is the bfs for that tableau.

We will discuss next lecture what to do if there is no obvious way of getting a tableau in canonical form.

For each constraint there is a basic variable

<b>z</b>	<b>x<sub>1</sub></b>	<b>x<sub>2</sub></b>	<b>x<sub>3</sub></b>	<b>x<sub>4</sub></b>		
<b>1</b>	<b>3</b>	<b>-2</b>	<b>0</b>	<b>0</b>	=	<b>0</b>
<b>0</b>	<b>-3</b>	<b>3</b>	<b>1</b>	<b>0</b>	=	<b>6</b>
<b>0</b>	<b>-4</b>	<b>2</b>	<b>0</b>	<b>1</b>	=	<b>2</b>

Objective function.  
Constraint 1  
Constraint 2

One **basic variable** is z

Constraint 1: **basic variable** is x<sub>3</sub>

Constraint 2: **basic variable** is x<sub>4</sub>

bfs

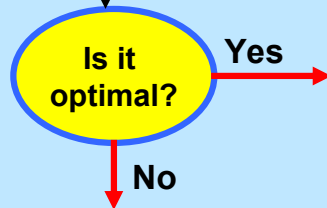
$$x_1 = 0; x_2 = 0;$$

$$x_3 = 6; x_4 = 2;$$

$$z = 0$$

## The simplex algorithm (for max problems)

Start with a feasible  
corner point solution



We were lucky to be able to start with a feasible bfs. We now move on to the rest of the algorithm.

Next lecture: how to find a starting bfs



## On the Optimality Conditions

<b>z</b>	<b>x<sub>1</sub></b>	<b>x<sub>2</sub></b>	<b>x<sub>3</sub></b>	<b>x<sub>4</sub></b>		
<b>1</b>	<b>3</b>	<b>-2</b>	<b>0</b>	<b>0</b>	=	<b>0</b>
<b>0</b>	<b>-3</b>	<b>3</b>	<b>1</b>	<b>0</b>	=	<b>6</b>
<b>0</b>	<b>-4</b>	<b>2</b>	<b>0</b>	<b>1</b>	=	<b>2</b>

The cost-coefficient of  $x_2$  is -2.

$$z + 3x_1 - 2x_2 = 0$$

The current bfs can be improved if we can increase  $x_2$  and hold  $x_1$  at 0.

If  $x_1 = 0$ , and  $x_2 = 1$ , then  $z = 2$ .

$$z + 3x_1 - 2x_2 = 0.$$

We can find a better solution by increasing  $x_2$  above 0 and adjusting the current basic variables to get a feasible solution.

z	$x_1$	$x_2$	$x_3$	$x_4$
1	3	-2	0	0
0	-3	3	1	0
0	-4	2	0	1

=

0
---

=

6
---

=

2
---

But won't we lose feasibility if we increase  $x_2$ ?

If increasing  $x_2$  improves the objective function, let's make it as large as we can!

Tim, the turkey

Cleaver, and MIT  
Beaver

42

Cleaver and Tim come right to the key issues.

## The Simplex Pivot

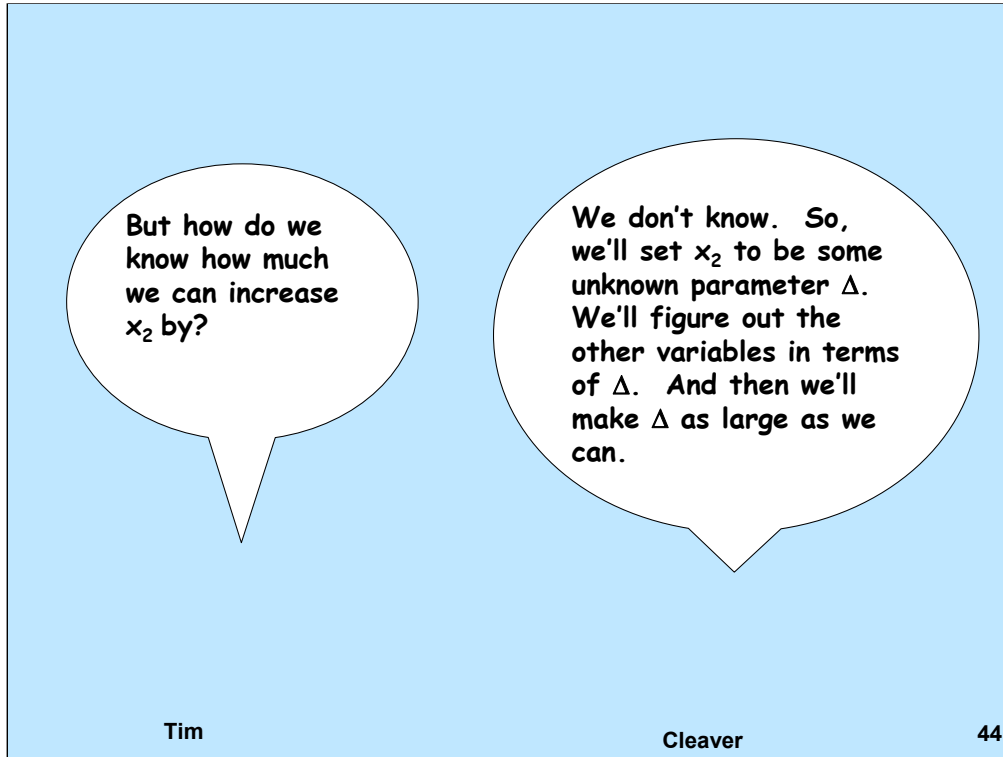
z	$x_1$	$x_2$	$x_3$	$x_4$	
1	3	-2	0	0	= 0
0	-3	3	1	0	= 6
0	-4	2	0	1	= 2

The way to do it is to increase  $x_2$  while simultaneously modifying basic variables to maintain feasibility. It's simple, but very clever.

Clever

43

I like Cleaver's enthusiasm for this material.



Tim is always asking good questions, even if he doesn't know many of the answers.

The current basic feasible solution (bfs) is not optimal!

<b>z</b>	<b>x<sub>1</sub></b>	<b>x<sub>2</sub></b>	<b>x<sub>3</sub></b>	<b>x<sub>4</sub></b>	
<b>1</b>	3	-2	0	0	= 0
0	-3	3	<b>1</b>	0	= 6
0	-4	2	0	<b>1</b>	= 2

$$x_2 = \Delta$$

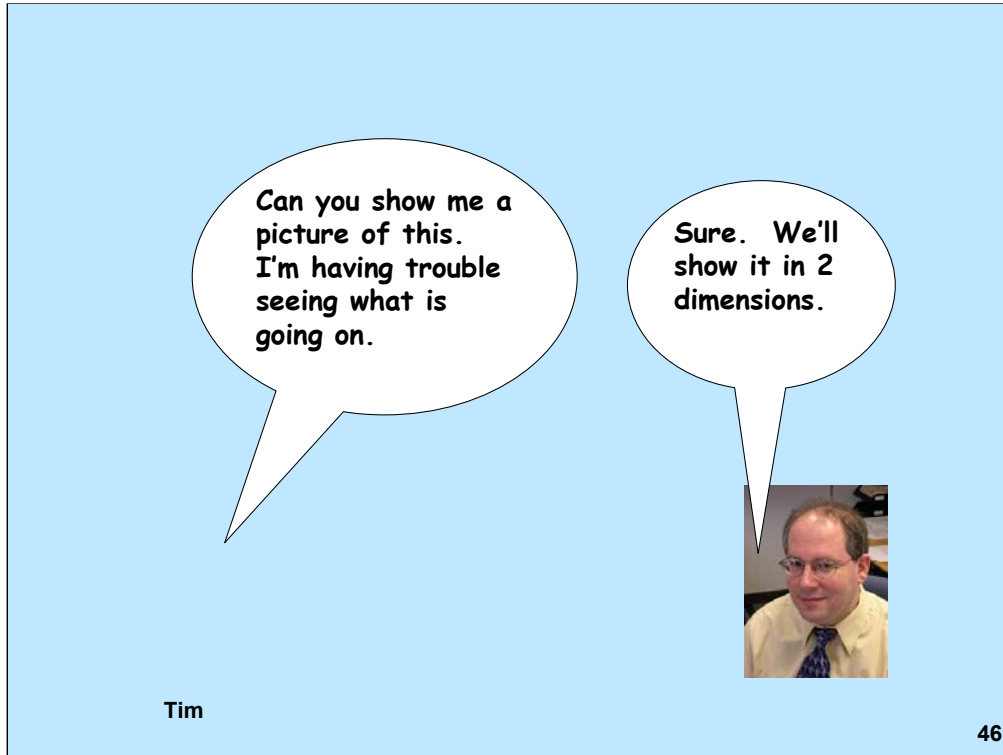
$x_1 = 0$ ,  
because we  
don't change  
any other non-  
basic variable.

$$\begin{aligned} z &= 2 \Delta. \\ x_3 &= 6 - 3 \Delta. \\ x_4 &= 2 - 2 \Delta. \end{aligned}$$

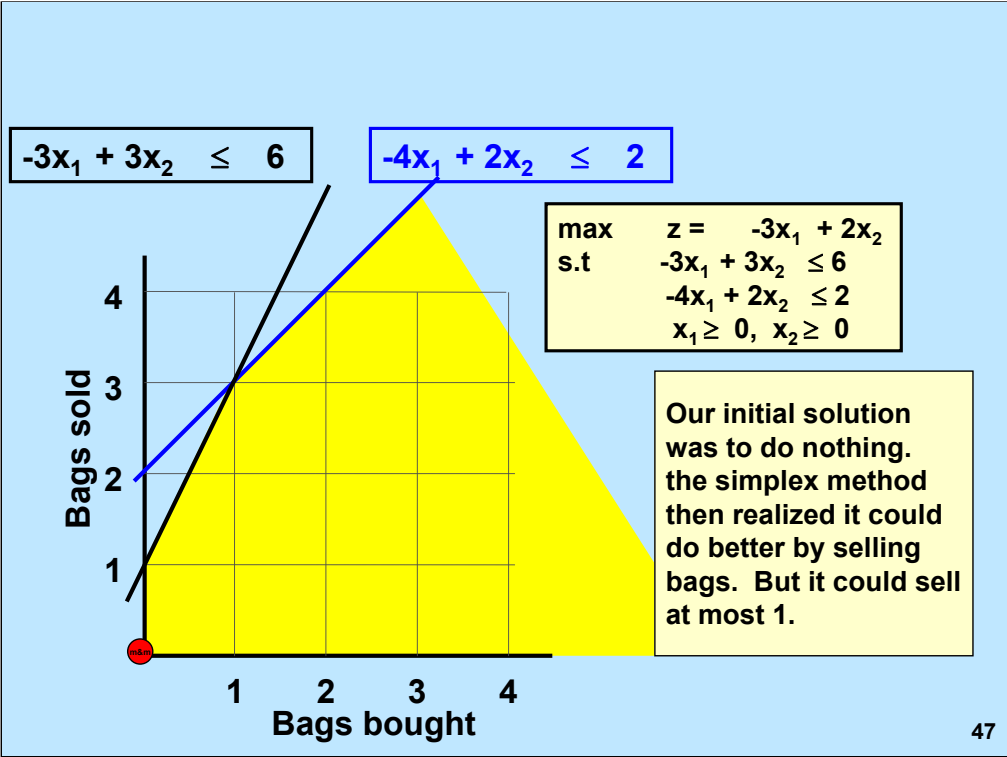
Choose  $\Delta$  as large as it can be so that all variables remain non-negative. That is, the solution stays feasible.

$$\Delta = 1$$

$$z = 2, x_1 = 0, x_2 = 1, x_3 = 3, x_4 = 0.$$



Occasionally, I put myself into the lectures as well.



**Note that the solution  $x_1 = 0$   
and  $x_2 = 1$  is a corner point.  
It turns out that it is also a  
basic feasible solution.**

**Ollie,  
the computationally  
wise owl.**

**48**

All bfs's correspond to corner point solutions. Ollie knew that, but decided to only tell you about a specific solution.



## Pivoting to obtain the bfs

z	$x_1$	$x_2$	$x_3$	$x_4$	
1	3	-2	0	0	= 0
0	-3	3	1	0	= 6
0	-4	2	0	1	= 2

Non-basic variable  $x_2$  becomes basic.

Choose column 2.

Basic variable  $x_4$  becomes non-basic.

$$z = 2, x_1 = 0, x_2 = 1, x_3 = 3, x_4 = 0.$$

Next iteration, we want the column of  $x_2$  to be

0

0

1

Since  $x_2$  replaces  $x_4$ , the column for  $x_2$  after the iteration (pivot) will be the same as the column for  $x_4$  before the iteration (pivot). In that way, we will still have canonical form after the pivot.

## Pivoting to obtain a better solution

<b>z</b>	<b>x<sub>1</sub></b>	<b>x<sub>2</sub></b>	<b>x<sub>3</sub></b>	<b>x<sub>4</sub></b>
<b>1</b>	<b>-1</b>	<b>0</b>	<b>0</b>	<b>1</b>
<b>0</b>	<b>3</b>	<b>0</b>	<b>1</b>	<b>-1.5</b>
<b>0</b>	<b>-2</b>	<b>1</b>	<b>0</b>	<b>.5</b>

**New Solution: basic variables z, x<sub>2</sub> and x<sub>3</sub>. Nonbasics: x<sub>1</sub> and x<sub>4</sub>.**

= **2**

= **3**

= **1**

**z = 2**  
**x<sub>1</sub> = 0**  
**x<sub>2</sub> = 1**  
**x<sub>3</sub> = 3**  
**x<sub>4</sub> = 0**

Note that the bfs after the pivot is exactly what we wanted. By letting  $x_2 = \Delta$  and increasing  $\Delta$  from 0 to 1, we were moving along an edge of the feasible region. At the end of the edge is another corner point.

## Summary of Simplex Algorithm

- **Start in canonical form with a basic feasible solution**
- 1. **Check for optimality conditions**
- 2. **If not optimal, determine a non-basic variable that should be made positive**
- 3. **Increase that non-basic variable, and perform a pivot, obtaining a new bfs**
- 4. **Continue until optimal (or unbounded).**

z	x <sub>1</sub>	x <sub>2</sub>	x <sub>3</sub>	x <sub>4</sub>	
1	a	0	0	0	= 3
0	b	0	1	0	= 6
0	c	1	0	0	= 3
0	d	0	0	1	= 5

To do with  
your partner  
(2 minutes)

The values a,  
b, c, and d are  
unknown

1. What are the basic variables? What is the current bfs?
2. Under what condition is the current bfs optimal?

z	$x_1$	$x_2$	$x_3$	$x_4$	
1	a	0	0	0	= 3
0	b	0	1	0	= 6
0	c	1	0	0	= 3
0	d	0	0	1	= 5

To do with  
your partner  
(3 minutes)

1. If we set  $x_1$  to  $\Delta$ , what are  $x_2$ ,  $x_3$ , and  $x_4$ , all expressed in terms of  $\Delta$ .
2. Assume that  $b > 0$  and  $d < 0$ . Under what condition we will set  $\Delta = 3/c$ ?
3. If  $\Delta = 3/c$ , what coefficient do we pivot on next?

## Recognizing Unboundedness

If the non-cost coefficients in the entering column are  $\leq 0$ , then the solution is unbounded

z	$x_1$	$x_2$	$x_3$	$x_4$		
1	-1	0	0	1	=	2
0	-3	0	1	-1.5	=	3
0	-2	1	0	.5	=	1

$z - x_1 + x_4 = 2$

$$z = 2 + \Delta$$

$$x_1 = \Delta$$

$$x_2 = 1 + 2\Delta$$

$$x_3 = 3 + 3\Delta$$

$$x_4 = 0$$

$\Delta$  can grow to  $\infty$ , and then  $z$  goes to  $\infty$ .

Next: two more iterations.

z	x <sub>1</sub>	x <sub>2</sub>	x <sub>3</sub>	x <sub>4</sub>		
1	-1	0	0	1	=	2    z - x <sub>1</sub> + x <sub>4</sub> = 2
0	3	0	1	-1.5	=	3    z = 2 + Δ
0	-2	1	0	.5	=	1    x <sub>1</sub> = Δ
						x <sub>2</sub> = 1 + 2Δ
						x <sub>3</sub> = 3 - 3Δ.
						x <sub>4</sub> = 0

The cost coefficient of x<sub>1</sub> in the z-row is negative.  
 Set x<sub>1</sub> = Δ and x<sub>4</sub> = 0.  
 Then Δ = 3/3.

## Another pivot

z	x <sub>1</sub>	x <sub>2</sub>	x <sub>3</sub>	x <sub>4</sub>		
1	0	0	+1/3	+1/2	=	3
0	1	0	1/3	-1/2	=	1
0	0	1	2/3	-1/2	=	3

z = 3
x <sub>1</sub> = 1
x <sub>2</sub> = 3
x <sub>3</sub> = 0
x <sub>4</sub> = 0

The largest value of  $\Delta$  is  $3/3$ .

Variable  $x_1$  becomes **basic**,  $x_3$  becomes **nonbasic**.

So,  $x_1$  becomes the basic variable for constraint 1.

Pivot on the coefficient with a 3.



### Check for optimality

$$z + x_3/3 + x_4/2 = 3$$

z	x <sub>1</sub>	x <sub>2</sub>	x <sub>3</sub>	x <sub>4</sub>		
1	0	0	+1/3	+1/2	=	3
0	1	0	1/3	-1/2	=	1
0	0	1	2/3	1/2	=	3

z = 3  
x<sub>1</sub> = 1  
x<sub>2</sub> = 3  
x<sub>3</sub> = 0  
x<sub>4</sub> = 0

**There is no negative coefficient in the z-row.**

**The current basic feasible solution is optimal!**

## Two views of the simplex method

- Improvement by “moving along an edge.”
  - Increase  $\Delta$ , and increase  $z$ .
  - An approach used in other algorithms, and that shows what is going on.
- Improvement by “moving to an adjacent corner point”
  - Move to an adjacent corner point and increase  $z$
  - It can be viewed as a “shortcut”

## Summary of Simplex Algorithm Again

- Start in canonical form with a basic feasible solution
- 1. Check for optimality conditions
  - Is there a negative coefficient in the cost row?
- 2. If not optimal, determine a non-basic variable that should be made positive
  - Choose a variable with a negative coef. in the cost row.
- 3. Increase that non-basic variable, and perform a pivot, obtaining a new bfs (or unboundedness)
  - We will review this step, and show a shortcut
- 4. Continue until optimal (or unbounded).

**The Minimum Ratio Rule for determining the leaving variable.**

<b>z</b>	<b>x<sub>1</sub></b>	<b>x<sub>2</sub></b>	<b>x<sub>3</sub></b>	<b>x<sub>4</sub></b>	=	
<b>1</b>	-3	0	0	0	=	3
0	3	1	0	0	=	6
0	-2	0	1	0	=	1
0	2	0	0	1	=	5

$z - 3x_1 = 3$

$z = 3 + 2\Delta$
$x_1 = \Delta$
$x_2 = 6 - 3\Delta$
$x_3 = 1 + 2\Delta$
$x_4 = 5 - 2\Delta$

$\Delta = \min (6/3, 5/2)$ . At next iteration, pivot on the 3.

ratio: RHS coefficient/ entering column coefficient

s.t. entering column coefficient is positive

## More on performing a pivot

- To determine the column to pivot on, select a variable with a negative cost coefficient
- To determine a row to pivot on, select a coefficient according to a minimum ratio rule
- Carry out a pivot as one does in solving a system of equations.

## **Next Lecture: More on the Simplex Algorithm**